

# Music training causes long-term enhancement of preschool children's spatial-temporal reasoning

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*Predictions from a structured cortical model led us to test the hypothesis that music training enhances young children's spatial-temporal reasoning. Seventy-eight preschool children participated in this study. Thirty-four children received private piano keyboard lessons, 20 children received private computer lessons, and 24 children provided other controls. Four standard, age-calibrated, spatial reasoning tests were given before and after training; one test assessed spatial-temporal reasoning and three tests assessed spatial recognition. Significant improvement on the spatial-temporal test was found for the keyboard group only. No group improved significantly on the spatial recognition tests. The magnitude of the spatial-temporal improvement from keyboard training was greater than one standard deviation of the standardized test and lasted at least one day, a duration traditionally classified as long term. This represents an increase in time by a factor of over 100 compared to a previous study in which listening to a Mozart piano sonata primed spatial-temporal reasoning in college students. This suggests that music training produces long-term modifications in underlying neural circuitry in regions not primarily concerned with music and might be investigated using EEG. We propose that an improvement of the magnitude reported may enhance the learning of standard curricula, such as mathematics and science, that draw heavily upon spatial-temporal reasoning. [Neurol Res 1997; 19: 2-8]*

*Keywords: Columnar cortical model; piano keyboard lessons; computer lessons; spatial recognition; educational implications; EEG*

## INTRODUCTION

Theoretical and empirical reports have suggested a relationship between musical and spatial reasoning abilities<sup>1-6</sup>. Leng and Shaw's model provides a neurobiological argument for a causal link between music and spatial-temporal reasoning<sup>7</sup>. Based on Mountcastle's<sup>8</sup> columnar<sup>8-9</sup> organizational principle for cortical function, the trion model<sup>10-12</sup> proposes that the inherent spatial-temporal firing patterns of highly structured, interconnected groups of neurons have the built-in ability to recognize, compare and find relationships among patterns<sup>12</sup>. This neural process may be responsible for the performance of spatial recognition tasks, such as classifying and recognizing physical similarities among objects. According to the model<sup>7</sup>, the evolution of these relationships among neural firing patterns into specific temporal sequences for tens of seconds over large portions of cortex allows for the performance of other more complex spatial tasks requiring spatial-temporal reasoning. Spatial-temporal reasoning involves

maintaining and transforming mental images in the absence of a physical model and is required for higher brain functions such as chess, mathematics and engineering.

Music cognition, it was argued, should also require these temporal sequences of neural activity<sup>7,13-14</sup>. A fundamental property of these evolving patterns of neural activity is that they can be readily strengthened through experience or learning<sup>10-11</sup>. Although higher brain functions are typically associated with specific, localized regions of cortex, all higher cognitive abilities draw upon a wide range of cortical areas<sup>15</sup>. Leng and Shaw<sup>7</sup> proposed that exposure to music might excite and enhance the cortical firing patterns used in spatial-temporal reasoning, thus affecting cognitive ability in tasks that share this complex spatial-temporal neural code. Behavioral research based on these predictions found that college students scored significantly higher on spatial-temporal reasoning tasks after listening to a Mozart sonata (K. 448), but not after listening to silence or to minimalist music<sup>16-17</sup>. While these studies established the existence of a causal relationship between music and spatial-temporal reasoning, the effect lasted only ten minutes. Leng and Shaw suggested that music training of young children, whose cortices are

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highly plastic, should produce long-term enhancement of spatial-temporal reasoning<sup>7</sup>. The aim of the present study was to test this hypothesis.

Our unpublished pilot study (in 1993) supported this prediction. After nine months of weekly individual piano keyboard lessons, a group of three-year-old children, enrolled at a music school, improved on a spatial-temporal task significantly more than was predicted by age-standardized norms<sup>18</sup>. A second group of three-year-old children from disadvantaged families, enrolled in an inner-city daycare center, received 30 min group singing lessons daily for nine months. This group also improved on the spatial-temporal task significantly more than predicted by age-standardized norms. Neither group improved significantly on spatial recognition tasks<sup>18</sup>. Encouraged by these results, we began a study in which we provided the music training under controlled conditions.

## METHODS

### Subjects

A total of 111 children were initially recruited. Thirty-three children withdrew from the preschools during the course of the study, and were not included in the analyses. The children who withdrew were fairly evenly distributed among the experimental groups. This left 78 children, 42 boys and 36 girls, for analysis. All children were of normal intelligence. The participants ranged in age from 3 years, 0 months to 4 years, 9 months at the start of the study. Three children were left-handed.

The study was conducted over a two year period using classes from three preschools. We provided 34 children, the Keyboard group, with private piano keyboard lessons and group singing sessions, and assigned the remaining children to one of three groups: Singing, Computer and No Lessons (*Figure 1*). The Singing group ( $n=10$ ) took part in the same singing activities as the

Keyboard group. The Computer group ( $n=20$ ) received private computer lessons matched in length and number to the piano keyboard lessons. The No Lessons group ( $n=14$ ) did not receive any training. None of the children had prior music lessons or computer lessons, and parental involvement was minimal.

All children in participating classrooms whose parents consented took part in the study. Children from the SA preschool were randomly assigned to either the Keyboard or the Computer groups (see *Figure 1*). The logistics of classroom schedules influenced group assignment in the other two preschools. Because the preschools assigned children to classrooms according to age, the children in the No Lessons group were older than the children in the other three groups (4 years, 1 month to 4 years, 9 months vs. 3 years, 0 months to 3 years, 11 months respectively). This was necessary to keep classes intact, to optimize sample size, and to avoid singling out children for participation in one treatment group over another. It is important to note, however, that the children's test scores were standardized by age, and no significant differences in test scores were found between groups prior to training.

### Training

We provided piano keyboard lessons rather than lessons on some other instrument because the keyboard gives a visual linear representation of the spatial relationships between pitches. We felt that coupling visual information with aural information might assist the neural pattern development relevant to spatial-temporal operations. Further support for using the keyboard came from our pilot study. We had no information regarding other instruments. We recruited professional keyboard instructors from the Irvine Conservatory of Music to provide the lessons at the preschools using Yamaha Portasound PSS 190 keyboards arranged on child-size tables. The 10-min private keyboard lessons consisted of performance exercises derived by the fourth author from traditional approaches (*Figure 2*). The children studied pitch intervals, fine motor coordination, fingering techniques, sight reading, music notation and playing from memory. After six months, all the children were able to perform basic primer-level melodies and simple melodies by Beethoven and Mozart. The preschools reserved an hour each day for keyboard practice.

Because many classes in the three schools already included some group singing activities, the Singing group was included to standardize these activities and to determine if such singing instruction would produce an effect in the absence of piano keyboard instruction, as indicated by the pilot study. The 30-min singing sessions were held five days a week, led by a music instructor. Songs included popular children's tunes and folk melodies.

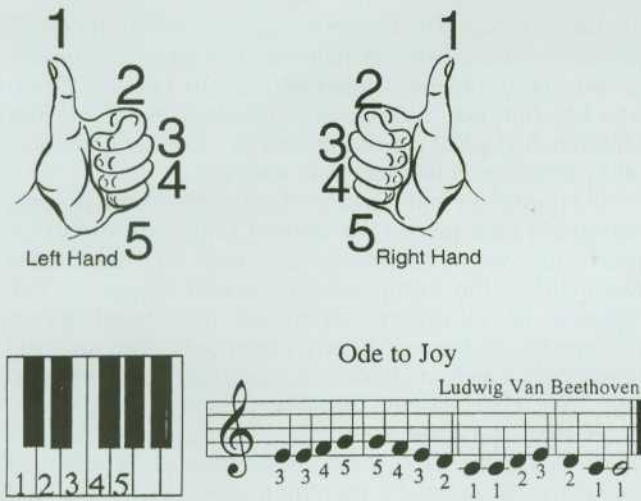
The Computer training controlled for the motor and visual coordination provided by the piano keyboard, personal attention, and the child's engagement with the activity. A professional computer instructor brought a

Lessons given at each preschool

School year	LB	WC	SA
1993-1994	Keyboard & No Lessons	Keyboard & No Lessons	—
1994-1995	Computer & Singing	Computer & Singing	Keyboard & Computer

**Figure 1:** Experimental design showing preschools and treatment groups. Seventy-eight children of diverse ethnicity and normal intelligence (42 males and 36 females) participated. The data were collected over two years from three preschools, LB, WC, and SA. SA participated in 1994-1995 only, and did not contribute to the No Lessons or Singing groups. No group differed significantly prior to training. LB and WC received lessons once per week for eight months; SA received lessons twice per week for six months. No effect was found for number of lessons ( $F_{(10,23)}=0.65$ , ns), so these data were pooled. For clarity, we henceforth refer to the interval between testing periods as six months





**Figure 2:** Numerical finger association used in the piano keyboard instruction. The children are taught to associate their fingers as numbered in **A** with the numbered keys on the piano keyboard as shown for the right hand in **B**. Shown in **C** is an example of a melody for the right hand that the children are able to play by the third month of lessons. Right and left hand coordination exercises are introduced at the outset of training. Hands together exercises are incorporated in the second and third months. Position changes or shifting to a different tonal center are introduced in the fourth and fifth months

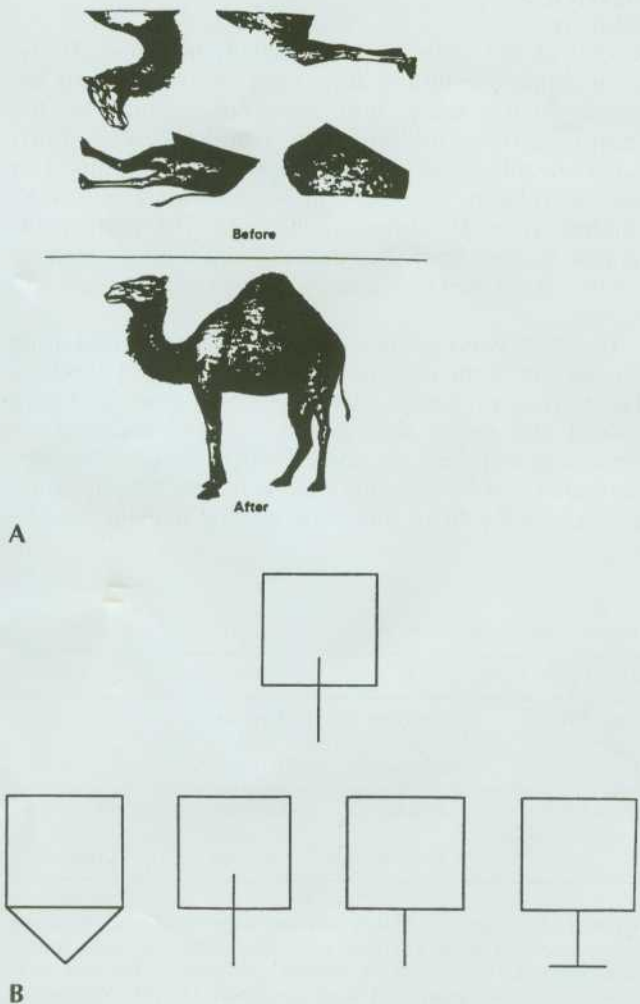
personal home computer to the preschools for the 10-min private lessons. The children were taught to open entertaining, age-appropriate, commercial software programs by copying simple DOS commands. Most of the children mastered this after one month. The software was designed to teach reading and simple arithmetic skills. Letter recognition varied for each student. Some children could identify many letters at the start, whereas most children could identify 8 to 10 letters after three months. The children also learned sentence structure by completing sentences such as 'I am thankful for ...'. Counting skills and number recognition were also taught. On average, children were able to count three objects after one month and six objects after three months. The lessons did not involve the use of the mouse or software programs which centrally featured music.

The No Lessons group controlled for task artifacts. For example, a particular task score may improve because the children enjoy it more with age, rather than as a function of treatment.

**Testing**

Prior to training, we tested all the children's spatial reasoning with four tasks from the Performance sub-test of the Wechsler Preschool and Primary Scale of Intelligence-Revised (WPPSI-R)<sup>18</sup>. Children were tested individually at the preschools. In the Object Assembly (OA) task, which measured spatial-temporal skill, the child arranged pieces of a puzzle to create a meaningful whole. Performing this task required forming a mental image of the completed object and rotating the puzzle pieces to match the image. Performance was facilitated

by putting the pieces together in particular orders, defining the spatial-temporal nature of the task. For example, in the animal puzzle (Figure 3A), beginning with the head facilitated performance, but placing the head and tail together first led to difficulty in correctly placing the middle pieces. The other five WPPSI-R tasks measured spatial-recognition. We used the following three: (i) Geometric Design required children to visually match and draw displayed model figures (Figure 3B), (ii) Block Design required the child to match depicted patterns using flat, two-colored blocks, and (iii) Animal Pegs required children to place the correctly colored pegs in holes below a series of pictured animals. These spatial-recognition tasks required matching, classifying, and recognizing similarities among displayed objects. Sequential order was not relevant. Children were re-tested on all tasks after six to eight months of lessons. The No Lessons group was re-tested at the same time as the other children.



**Figure 3 A:** Schematic representation of the Object Assembly task, requiring spatial-temporal reasoning. The child arranges pieces of a puzzle to create a meaningful whole. **B:** Schematic representation of the Geometric Design task, requiring spatial recognition. The child points to the bottom-row figure that matches the figure in the top row



**Scoring**

As specified by the WPPSI-R scoring instructions<sup>18</sup>, raw scores were based on the number of errors made within a specified time period, and bonus points were awarded for accuracy and speed. Scaled scores were calculated for children at three-month age intervals. The established mean for all WPPSI-R tests (*M*) is 10 points, with all standard deviations (*σ*) equal to 3.

Testing procedures followed those recommended by the Wechsler test manual<sup>18</sup>. Testing sessions lasted 60–75 min, and were conducted in the morning. Children who became distracted during testing were given a 5-min break before testing was resumed. Testing during the first year was performed by the first author. During the second year, testing was performed by research assistants blind both to the hypothesis of the experiment and to group assignment. Preliminary analysis conducted on these two sets of data showed no differences, so the data were pooled. All tasks were independently scored by two researchers blind to condition assignment. Inter-rater reliability ranged from *r* = 0.995 to *r* = 1.0.

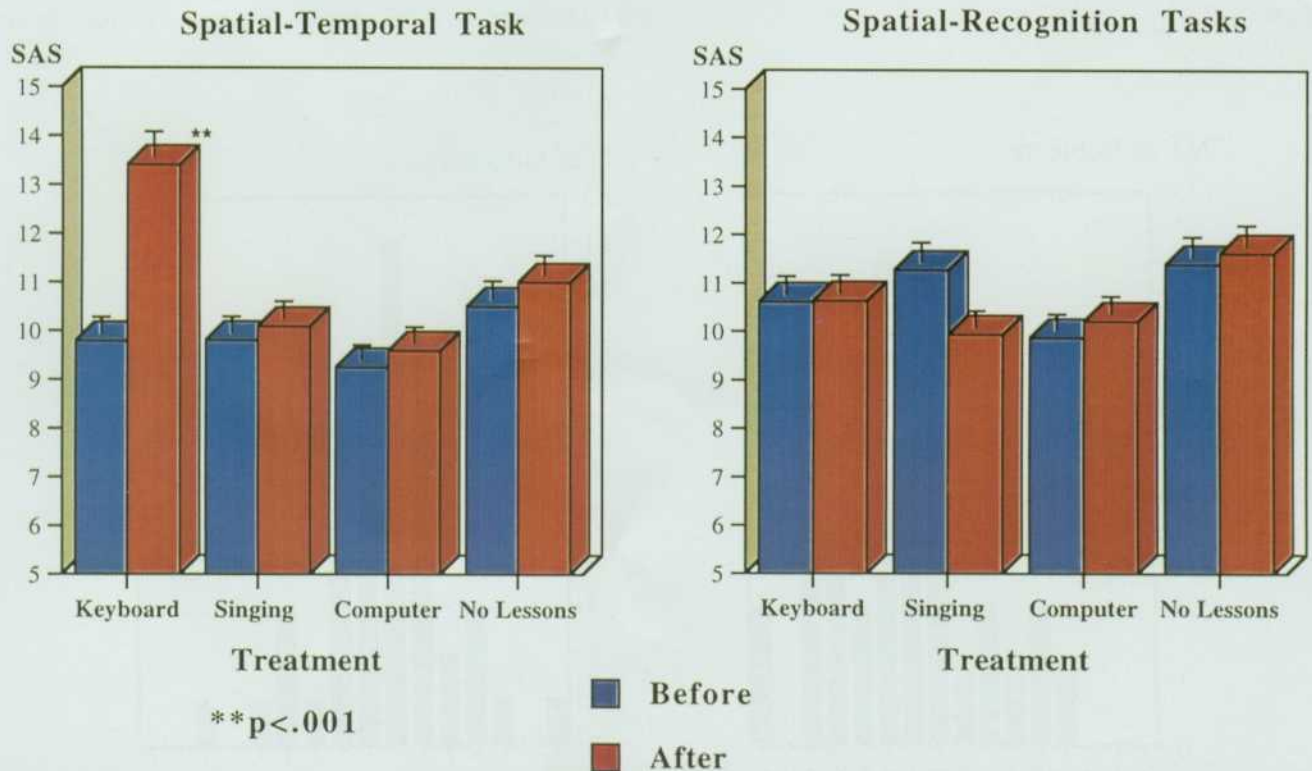
**RESULTS AND DISCUSSION**

Figure 4A shows how each of the four different types of training effect spatial-temporal abilities by presenting

the before and after training Object Assembly (OA) means for each of the four training groups. This figure reveals that music training for the Keyboard group produced a dramatic overall increase in OA scores (as evidenced by a pre-training mean of 9.79 and a post-training mean of 13.41) while none of the other training groups showed any appreciable change. To verify the obvious difference, a One-Way ANOVA was performed on the change scores with the four training groups (Keyboard, Computer, Singing, No Lessons) as treatments. As expected, this analysis produced highly significant differences between the four groups ( $F(3,74) = 3.87, p < 0.0001$ ).

Even more revealing were the results derived from a subsequent assessment of multiple paired comparisons between treatment groups where Bonferroni (Dunn) T tests were used to assess differences between means of pre-post difference scores for pairs of treatment groups. Using this conservative method, the Keyboard group nonetheless differed significantly ( $p < 0.01$ ) from each of the other three groups. No pairing of the remaining three treatment groups produced a rejection of the associated null hypothesis, even when the alpha level was set as high as 0.99 ( $\alpha = 0.99$ ). As shown in Table 1, the pattern of these results is quite striking.

ANOVAs performed on the children's scores on the other tests (Geometric Design, Block Design and Animal



**Figure 4 A:** The means for the OA standard age scores (SAS) with standards errors are plotted measuring spatial-temporal reasoning for the Keyboard, Singing, Computer and No Lessons groups before and after treatment. The standard deviations for each of these groups, before and after training, respectively, are 2.61 and 2.91, 3.91 and 3.07, 3.27 and 3.07, 2.77 and 2.25. The Keyboard group improved significantly following lessons whereas the other groups did not. **B:** The spatial-recognition SAS means with standard errors for the four groups before and after treatment. The standard deviations for each of these groups, before and after training, respectively, are 2.25 and 1.86, 1.68 and 1.97, 2.53 and 2.82, 1.55 and 1.57. No groups improved significantly

Pegs), were not significant, indicating that the children's scores on the items which measured spatial-recognition did not improve significantly after lessons (Figure 4B).

The OA scores of the Singing, Computer and No Lessons groups did not improve significantly, nor did their scores on the spatial-recognition tasks. The lack of

significant improvement of the Singing group on the OA task (9.80 before and 10.10 after) suggests that either a more structured singing program is required or that experience with a musical instrument, with its visual and motor representation of spatial-temporal relations between sequences of pitches, may be crucial to the effect. We cannot rule out the possibility, however, that the singing lessons contributed to the enhancement of spatial-temporal reasoning found for the Keyboard group. We suspect that the significant improvement found in the pilot study by the inner-city children who received group singing lessons may have been due to the school's demographic composition. Clearly, the effects of demographics need to be investigated.

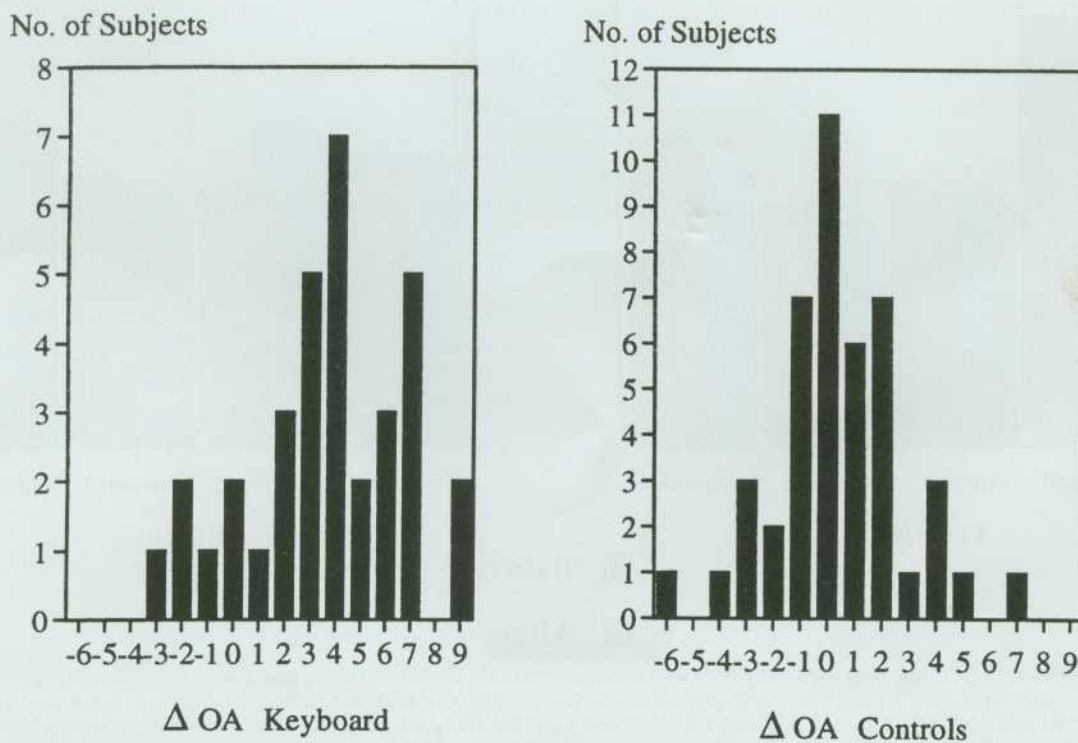
The stability of the Computer group's OA scores (9.25 before lessons vs. 9.60 after lessons), rules out attention, motivation and motor coordination as primary contributors to the Keyboard group's improvement. This, we believe, is a substantial finding given the captivating nature of the animated images used in the computer training. Finally, the OA scores of the No Lessons group did not improve significantly after eight months ( $M=10.50$  before vs. 11.00 after). This indicates that the Keyboard group's improvement was not due to task artifacts.

To convey the extent of the Keyboard group's improvement on the OA task, we plotted the histogram of SAS after treatment minus SAS before treatment ( $\Delta OA$ ) for the Keyboard group (Figure 5A). Twenty-four of the 34 children, or 71%, improved by 3 or more points

**Table 1:** Bonferroni (Dunn) T tests for variable: OA(post) - OA(pre)

Condition comparison	Simultaneous lower confidence limit	Difference between means	Simultaneous upper confidence limit
Keyboard - No Lessons	0.2662	3.1176	5.9691 *
Keyboard - Computer	0.7373	3.2676	5.7980 *
Keyboard - Singing	0.0875	3.3176	6.5478 *
No Lessons - Computer	-2.9790	0.1500	3.2790
No Lessons - Singing	-3.5178	0.2000	3.9178
Computer - Singing	-3.4277	0.0500	3.5277

This test controls the type I experiment wise error rate but generally has a higher type II error rate than Tukey's for all pairwise comparisons. Alpha = 0.01; Confidence = 0.99; df = 74; MSE = 7.569992; Critical Value of T = 3.26358; Comparisons significant at the 0.01 level are indicated by \*



**Figure 5 A:** Histogram of SAS for OA after treatment minus SAS before treatment ( $\Delta OA$ ) for the 34 children in the Keyboard group. **B:** The corresponding histogram for the 44 children in the combined three control groups



(WPPSI-R's  $\sigma = 3$ ), as compared to the expected 5 or 6 children (16%) by the Gaussian model. Figure 5B shows the histogram for the three combined control groups. Only six of the 44 children (14%) improved by 3 or more points.

Memory researchers differentiate between short-term and long-term memory<sup>19</sup>. The latter, lasting hours or longer, is associated with enduring synaptic changes, perhaps long-term, potentiation<sup>20-21</sup>. To determine if the enhancement found in this study was long-term, we compared the  $\Delta$ OA scores of the children who were tested one day or more after their last keyboard lesson to those of the children who were tested less than one day afterwards. We found no significant difference. A *t*-test for independent samples performed on the  $\Delta$ OA scores of the 27 children who were tested one day or more after their last piano keyboard lesson ( $M = 3.59$ ) vs. the  $\Delta$ OA scores of the 7 children who were tested less than one day afterwards ( $M = 3.71$ ) was not significant ( $t_{(1,32)} = 0.09$ , ns). This indicates that the enhancement on the OA spatial-temporal task from piano keyboard training lasted at least one day, and is considered by memory researcher standards to be long-term<sup>19-21</sup>.

Our previous findings of enhanced short-term spatial-temporal reasoning in college students after listening to a Mozart sonata suggest that music can prime regions of cortex responsible for spatial-temporal reasoning<sup>16-17</sup>. (An EEG coherence study of this short-term enhancement of spatial-temporal reasoning has been performed<sup>22</sup>.) The long-term enhancement found in the current study represents an increase by more than a factor of 100 over the previous listening experiments<sup>16-17</sup>. This study suggests that music training, unlike listening, produces long-term modifications in underlying neural circuitry (perhaps right prefrontal and left temporal cortical areas as indicated in EEG coherence studies<sup>22</sup>) in regions not primarily concerned with music. The magnitude of the improvement in spatial-temporal reasoning from music training was greater than one standard deviation, equivalent to an increase from the 50th percentile on the WPPSI-R standardized test to above the 85th percentile.

The precise duration of the enhancement and the possible existence of a critical period need to be examined. An exploration of the aspects of music training that are responsible for the enhancement must be undertaken, so that the optimum training method can be identified. Although our study was limited by the resources we had available, the ideal study would draw participants from the same preschool at the same time, thereby eliminating any possible confounds due to preschool demographics or age. It should be noted, however, that we found no significant differences on measures of task improvement based on these factors. Further research is necessary to identify other spatial-temporal reasoning tasks that may be enhanced by music training. And finally, explorations into the cortical representation<sup>22</sup> of spatial-temporal and musical reasoning coupled with supporting behavioral data are necessary.

It has been clearly documented<sup>23</sup> that young students

have difficulty understanding the concepts of proportion (heavily used in math and science) and that no successful program has been developed to teach these concepts in the school system. We predict that an enhanced ability to evolve temporal sequences of spatial patterns as a result of music training will lead to an enhanced conceptual mastering of proportional reasoning. This is a strong proposal which should be investigated in future research.

The high proportion of children who evidenced this dramatic improvement in spatial-temporal reasoning as a result of music training (Figure 5A) should be of great interest to scientists and educators, particularly because the duration of the effect lasts at least one day. We suggest that an improvement of this magnitude may enhance the learning of standard school curricula that draw heavily upon spatial-temporal reasoning abilities, such as mathematics and science.

#### ACKNOWLEDGEMENTS

This research was supported by grants from the National Association of Music Merchants, the Ralph and Leona Gerard Foundation, The Seaver Institute, the Orange County Philharmonic Society, Walter Cruttenden and Associates, the National Academy of Recording Arts and Sciences and the National Piano Foundation. The Yamaha Corporation of America supplied the keyboards. We thank the preschool directors J. Kolinsky, G. Morgan-Beazell, M. Rice and D. Rippetoe and their staffs for their patience, cooperation and support. We thank B. Grice of KUSC Radio and J. Fuerbringer of the O.C. Philharmonic Society for help in recruiting the preschools. We are indebted to keyboard instructors C. Jones, D. Schultheiss, and M. Wright; computer instructor T. Earl; singing instructors T. Lundeen, L. Mendoza, M. Navarro and R. Wise; testers L. Cheung, L. Goldhammer, S. Jethwa and J. Johnson; and laboratory assistant K. Gitchoff for their invaluable contributions. Finally, we are grateful to L. Brothers, D. Dooley, J. McGaugh, G. Palm and C. Stephens for reviewing earlier drafts of this manuscript, and to K. Bruhn and J. Kabakov for their enthusiastic encouragement.

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